Advanced ODE-Lecture 4
Continuous and Differentiable Solution on Data

Dr. Zhiming Wang

Professor of Mathematics
East China Normal University, Shanghai, China

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- **Continuous Dependence of Solution on Data**
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Motivation

1) Continuous Dependence on Data

- A good math model should have continuous solutions w.r.t. an initial data (t_0, x_0) and a system parameter μ of $f(t, x, \mu)$ - small errors in data or parameter yield solutions that are close (over some **finite** time interval) – **Well-Posedness!**

- The above property is called continuous dependence on data (parameter). This continuous dependence property is not possible at points where the solution is not unique! **Why?**

2) Sensitivity of Variation on Data

- It is natural to ask differentiability of solutions w.r.t. data to characterize the sensitivity of variation on data – **Differentiability Theorem**.

Remark 4.1. The general form of the IVP with data is given by

$$\begin{cases} x' = f(t, x, \mu) \\ x(t_0) = x_0 \end{cases}, \qquad (E_\mu)$$

where $\mu = (\mu_1, \mu_2, \cdots, \mu_s)^T \in R^s$. Let $z = (x, \mu) \in R^n \times R^s$, then

$$\begin{cases} z' = \overline{f}(t, z) \\ z(t_0) = z_0 \end{cases} \Leftrightarrow \begin{cases} x' = f(t, x, \mu), & \mu' = 0 \\ x(t_0) = x_0, & \mu(t_0) = \mu \end{cases}. \qquad (H_\mu)$$

It is easy to show (**Homework-1**):

a) $z(t) = (x(t), \mu(t))$ is a solution of (H_μ) $\Leftrightarrow$ $x(t)$ is a solution of (E_μ) and $\mu(t) \equiv \mu$;

b) If (E_μ) has a unique solution, so does (H_μ).

Then (H_μ) has the same structure to the IVP given before. The only difference is their dimensions. For simplicity of notation, we still consider the IVP just regarding (t_0, x_0) as data.

Continuous Dependence of Solution on Data

1) Well-Posedness. The IVP is called **well posed** if there exists a unique solution $x(t, t_0, x_0)$, which depends continuously on (t_0, x_0).

2) Continuous Dependence of Solution on Data (t_0, x_0)

The real initial value (t_0, x_0) is obtained by measurement. Suppose the measured initial value is (t_0, x_0), satisfying the following error condition.

$$|t_0 - t_0^0| \le \frac{h}{2}; \quad \| x_0 - x_0^0 \| \le \frac{b}{2},$$

where (t_0^0, x_0^0) is the nominal (ideal) initial value such that the following IVP

$$\begin{cases} x' = f(t, x) \\ x(t_0^0) = x_0^0 \end{cases} \qquad (E_0)$$

has a unique solution $x(t, t_0^0, x_0^0)$ in

$$Q = \{(t, x) \in R \times R^n : |t - t_0^0| \le a, \ \| x - x_0^0 \| \le b\}.$$

Let $U=\{(t_0,x_0)\in R\times R^n:|t_0-t_0^0|\le\frac{h}{2},\|x_0-x_0^0\|\le\frac{b}{2}\}\subseteq Q$. Then, the continuous property of $x(t,t_0,x_0)$ of the IVP in the defined domain is discussed as follows.

$$G=\{(t,t_0,x_0)\in R\times R\times R^n:|t-t_0|\le\frac{h}{2};(t_0,x_0)\in U\}.$$

Theorem 4.1 Suppose that $f(t,x)$ is continuous; Lipschitz on Q and $(t_0,x_0)\in U$. Then the solution $x(t,t_0,x_0)$ of the IVP is continuous on $(t,t_0,x_0)\in G$.

Proof. First, we construct the Picard sequence $\{x_n(t,t_0,x_0)\}$ ($n\in N^+$) on $t\in[t_0-h,t_0+h]$ as follows.

$$x_0(t,t_0,x_0)=x_0,\quad (t,t_0,x_0)\in G;$$

$$x_1(t,t_0,x_0)=x_0+\int_{t_0}^{t}f(s,x_0(s,t_0,x_0))\,ds,\quad (t,t_0,x_0)\in G;$$

$$\cdots$$

$$x_{n+1}(t, t_0, x_0) = x_0 + \int_{t_0}^{t} f(s, x_n(s, t_0, x_0))ds, \quad (t, t_0, x_0) \in G;$$

$$\cdots$$

It is the same to show that the Picard sequence $\{x_n(t, t_0, x_0)\}$ is uniformly convergent to $x(t, t_0, x_0)$. That is,

$$\| x_n(t, t_0, x_0) - x(t, t_0, x_0) \| \le \frac{ML^{n-1}}{n!} | t - t_0 |^n \le \frac{ML^{n-1}}{n!} (| t - t_0^0 | + | t_0^0 - t_0 |)^n$$

$$\le \frac{ML^{n-1}}{n!} (\frac{h}{2} + \frac{h}{2})^n = \frac{ML^{n-1}}{n!} h^n.$$

Meanwhile, $x(t, t_0, x_0)$ is a solution of the IVP, and continuous on $(t, t_0, x_0) \in G$ because $x_n(t, t_0, x_0)$ is continuous on $(t, t_0, x_0) \in G$ for each $n \in N^+$. □

Remark 4.2 For each $n \in N^+$, $x_n(t, t_0, x_0)$ is continuous on (t_0, x_0) for the fixed $t \in [t_0^0 - \frac{h}{2}, t_0^0 + \frac{h}{2}]$. Therefore, $x_n(t, t_0, x_0)$ for each $n \in N^+$ is continuous on $(t, t_0, x_0) \in G$.

Remark 4.3 The reason of defining U and G above. Since $(t_0, x_0) \in U$ in $\{x_n(t, t_0, x_0)\}$, the interval $|t - t_0| \le h$ varies with t_0. Therefore, the intervals of $\{x_n(t, t_0, x_0)\}$ for each $n \in N^+$ may not be the same in general. For a rigorous sense, we may find

$$[t_0^0 - \frac{h}{2}, t_0^0 + \frac{h}{2}] = \bigcap_{|t_0 - t_0^0| \le \frac{h}{2}} [t_0 - h, t_0 + h]$$

that is a common interval of $\{x_n(t, t_0, x_0)\}$ by using $|t_0 - t_0^0| \le \frac{h}{2}$. Therefore, $t \in [t_0^0 - \frac{h}{2}, t_0^0 + \frac{h}{2}]$ is a reasonable common interval. That is, $(t, t_0, x_0) \in G$.

A Strong Version of Continuous Dependence on Data

Theorem 4.2 Suppose that G is an open set in $R \times R^n$, $f : G \to R^n$ is continuous and local Lipschitz. Let $x(t, t_0, x_0)$ be a solution of the IVP defined on $[t_0, b]$, $b < \omega_+$, and $x(t, t_0, x_1)$ be a solution of the following IVP:

$$\begin{cases} x' = f(t, x) \\ x(t_0) = x_1 \end{cases} . \qquad (E_1)$$

Then, for $\forall \varepsilon > 0$, there exists $\delta > 0$ s.t. $\| x_1 - x_0 \| < \delta \Rightarrow x(t, t_0, x_1)$ is also defined on $[t_0, b]$. Moreover, we have

$$\| x(t, t_0, x_1) - x(t, t_0, x_0) \| \le \varepsilon , \quad t \in [t_0, b] .$$

Proof. Choose $0 < \varepsilon << 1$ such that

$$K = \{(t, x) : t \in [t_0, b],\ \| x - x(t, t_0, x_0) \| \le \varepsilon\} \subset G .$$

Let L be Lipschitz on the compact set K. It yields that f satisfies

$$\| f(t, x_1) - f(t, x_0) \| \le L \| x_1 - x_0 \|, \text{ for } (t, x_0), (t, x_1) \in K .$$

Taking $(\infty >)\ \delta = \varepsilon e^{-L(b-t_0)} > 0$ since b is finite, we claim that the interval of existence for $x(t, t_0, x_1)$ in K must not be less than $[t_0, b]$. We show it by contradiction. If $x(t, t_0, x_1)$ is defined on $[t_0, \overline{b}]$ with $\overline{b} < b$, then one has

$$\| x(t, t_0, x_1) - x(t, t_0, x_0) \| \le \| x_1 - x_0 \| + L \int_{t_0}^{t} \| x(s, t_0, x_1) - x(s, t_0, x_0) \| \, ds$$

$$\Rightarrow \ \| x(t, t_0, x_1) - x(t, t_0, x_0) \| \le \| x_1 - x_0 \| e^{L(t-t_0)} \le \delta e^{L(t-t_0)} \le \varepsilon ,$$

whenever $\| x_1 - x_0 \| < \delta$ (Gronwall inequality)

$$\Rightarrow \quad \| x(\bar{b}, t_0, x_1) - x(\bar{b}, t_0, x_0) \| \le \| x_1 - x_0 \| e^{L(\bar{b}-t_0)} < \delta e^{L(\bar{b}-t_0)}$$

$$= \varepsilon e^{-L(b-t_0)} e^{L(\bar{b}-t_0)} = \varepsilon e^{L(\bar{b}-b)} < \varepsilon .$$

This shows that the point $(\bar{b}, x(\bar{b}, t_0, x_1))$ is an interior of K, which can be extended further by Picard theorem. This is contradictive to the assumption. Applying Gronwall inequality once more on $t \in [t_0, b]$ yields

$$\| x(t, t_0, x_1) - x(t, t_0, x_0) \| \le \| x_1 - x_0 \| e^{L(t-t_0)} \le \delta e^{L(t-t_0)} < \varepsilon ,$$

whenever $\| x_1 - x_0 \| < \delta$.

Therefore, $x(t, t_0, x_0)$ is a continuous function of x_0. □

Remark 4.4 In fact, $x(t, t_0, x_0)$ is also Lipschitz on x_0 because there exists a constant C if b is finite s.t.

$$\| x(t, t_0, x_0) - x(t, t_0, x_1) \| \le C \| x_0 - x_1 \|, \quad t \in [t_0, b].$$

However, such a Lipschitz constant $C = C(t) = e^{L(t-t_0)} \le e^{L(b-t_0)}$ depends on the finite interval of $[t_0, b]$ with $b < \infty$. If $b \to \infty$, this property is not true!!! See $\delta = \varepsilon e^{-L(b-t_0)} \to 0$ as $b \to \infty$. That is, for any given $\varepsilon > 0$, we can't find any finite $\delta > 0$ to have a desired property!!! Continuous dependence on data for $[t_0, \infty)$ is in fact a global issue that needs an additional condition for sure. It is referred to **Lyapunov stability theory**.

Remark 4.5 The IVP is always wellposed if f is continuous and locally Lipschitz on any finite time interval $[t_0, b]$ with $b < \infty$. However, under such conditions, the continuous dependence w.r.t data is not true on an infinite time interval $[t_0, \infty)$.

Differentiability of Solution on Data

Theorem 4.3 Suppose that $f(t,x)$ of the IVP is of C^2 on Q and $(t_0, x_0) \in U$. Then the solution $x(t, t_0, x_0)$ of the IVP is continuously differentiable on $(t, t_0, x_0) \in G$.

Proof. By Theorem 4.1, we take a Picard sequence

$$x_{n+1}(t, t_0, x_0) = x_0 + \int_{t_0}^{t} f(s, x_n(s, t_0, x_0))ds, \quad n \in N^+, \quad (t, t_0, x_0) \in G,$$

which is continuous on $(t, t_0, x_0) \in G$ for each $n \in N^+$ and uniformly convergent to the solution $x(t, t_0, x_0)$ of the IVP, which is continuous on $(t, t_0, x_0) \in G$.

Since $f'_x(t,x)$ is also continuously differentiable on Q, we construct an associated Picard matrix sequence as follows.

$$Y_{n+1}(t,t_0,x_0) = I_n + \int_{t_0}^{t} f'_x(s, x_n(s,t_0,x_0))\, Y_n(t,t_0,x_0)ds,$$

where $n \in N^+$ and $(t,t_0,x_0) \in G$. It is similar to show that $\{Y_n(t,t_0,x_0)\}$ is well defined, continuous on $(t,t_0,x_0) \in G$ and uniformly convergent to $Y(t,t_0,x_0)$ that is continuous on $(t,t_0,x_0) \in G$ (**Homework-2**).

Next, we remark $\dfrac{\partial x_0(t,t_0,x_0)}{\partial x_0} = I_n = Y_0(t,t_0,x_0)$. Then by the definitions of $\{x_n(t,t_0,x_0)\}$ and $\{Y_n(t,t_0,x_0)\}$, we conclude by induction on $n \in N^+$ that

$$\frac{\partial x_n(t,t_0,x_0)}{\partial x_0} = Y_n(t,t_0,x_0),\quad (t,t_0,x_0) \in G, \text{ for each } n \in N^+.$$

Therefore, $\{Y_n(t, t_0, x_0)\}$ is a derivative sequence of $\{x_n(t, t_0, x_0)\}$ w.r.t x_0. Since $\{x_n(t, t_0, x_0)\}$ and $\{Y_n(t, t_0, x_0)\}$ are both uniformly convergent, their limits are

$$\frac{\partial x(t, t_0, x_0)}{\partial x_0} = Y(t, t_0, x_0), \quad (t, t_0, x_0) \in G .$$

Then we conclude that $\dfrac{\partial x(t, t_0, x_0)}{\partial x_0}$ is continuous on $(t, t_0, x_0) \in G$.

It is similar to show that

$$\frac{\partial x_{n+1}(t, t_0, x_0)}{\partial t_0} = -f(t_0, x_0) + \int_{t_0}^{t} f_x'(s, x_n(s, t_0, x_0)) \frac{\partial x_n(t, t_0, x_0)}{\partial t_0} ds$$

is well defined, continuous and uniformly convergent on $(t, t_0, x_0) \in G$. Then we conclude that $\dfrac{\partial x(t, t_0, x_0)}{\partial t_0}$ is continuous on $(t, t_0, x_0) \in G$ (**Homework-3**). □

We have simultaneously proved the following theorem.

Theorem 4.4 Suppose that $f(t,x,\mu)$ of (E_μ) is of C^2 on $Q\times D_\mu$, That is, $f'_x(t,x,\mu)$ and $f'_\mu(t,x,\mu)$ are continuously differentiable in $Q\times D_\mu$. Then the solution $x(t,t_0,x_0,\mu)$ of (E_μ) is continuously differentiable on (t,t_0,x_0,μ) in some neighborhood. Moreover, $\dfrac{\partial x(t,t_0,x_0,\mu)}{\partial t_0}$, $\dfrac{\partial x(t,t_0,x_0,\mu)}{\partial x_0}$ and $\dfrac{\partial x(t,t_0,x_0,\mu)}{\partial \mu}$ are respectively the solutions of the following IVP

$$z' = f_x'(t,x(t,t_0,x_0,\mu))z, \quad z(t_0) = -f(t_0,x_0,\mu); \tag{F1}$$

$$z' = f_x'(t,x(t,t_0,x_0,\mu))z, \quad z(t_0) = I_n; \tag{F2}$$

$$z' = f_x'(t,x(t,t_0,x_0,\mu))z + f_\mu'(t,x(t,t_0,x_0,\mu)), \quad z(t_0) = O_{n\times s}. \tag{F3}$$

Proof. By Theorem 4.3, $x(t, t_0, x_0, \mu)$ is continuously differentiable on its arguments (t, t_0, x_0, μ). Then, we take derivatives on both side of

$$x(t, t_0, x_0, \mu) = x_0 + \int_{t_0}^{t} f(s, x(t, t_0, x_0, \mu), \mu) ds$$

to get (F1)-(F3) immediately. □

Remark 4.6 The IVP (F1)-(F3) are all linear systems that are seemly easy to be solved. However, they are only conceptual important because $f_x'(t, x(t, t_0, x_0, \mu))$ and $f_\mu'(t, x(t, t_0, x_0, \mu))$ are unknown without knowing the solution $x(t, t_0, x_0, \mu)$ in advance.

Remark 4.7 The condition f being C^2 both in Theorem 4.3 and Theorem 4.4 can be replaced by a mild condition of f being continuously differentiable. However, the present proof no longer workable and a more complicated method will be employed.

Summary

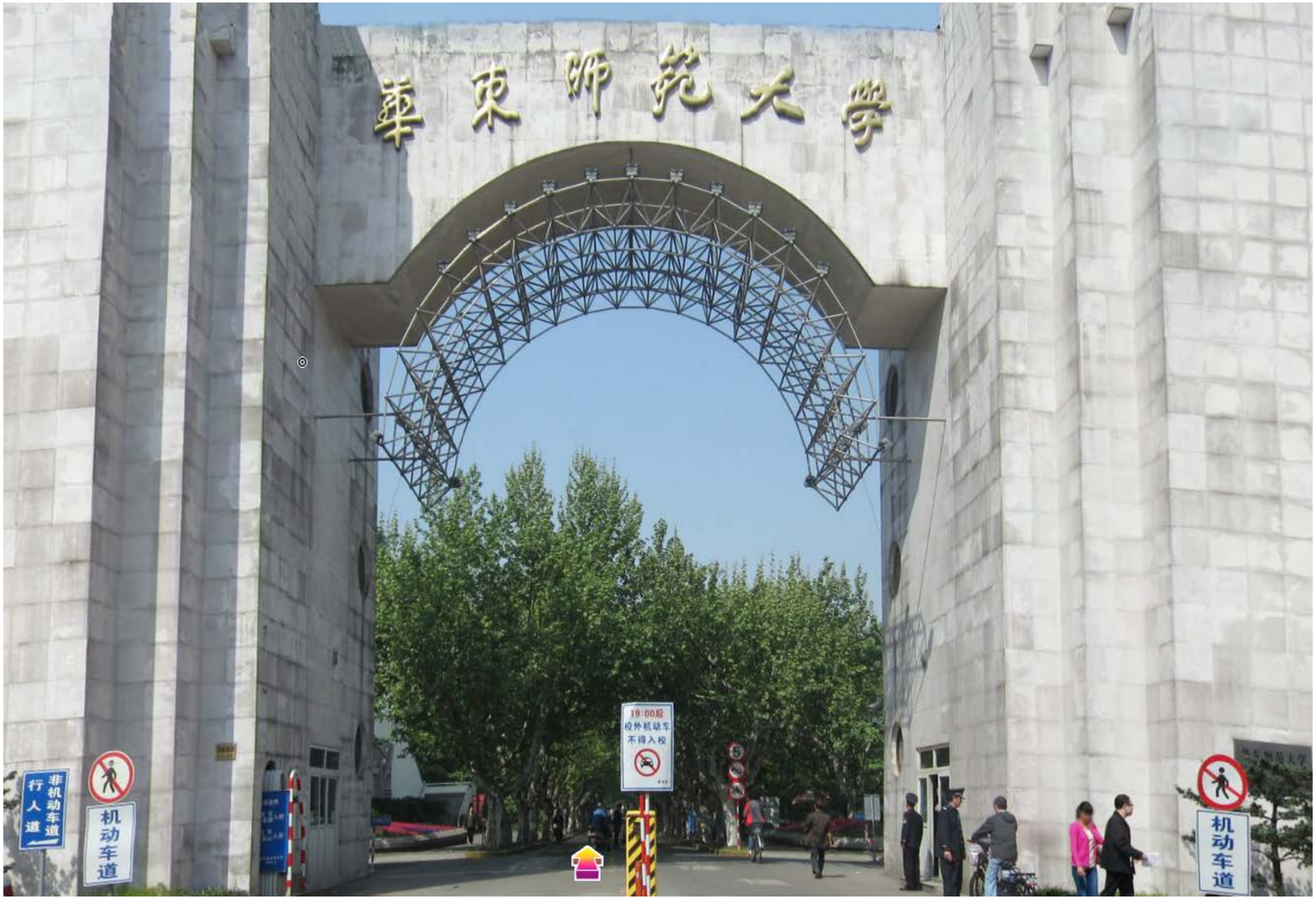
- Under mild conditions, the solutions depend continuously on the data for any finite closed interval.
- A good math model should have continuous dependence and differentiability on its data.
- Continuous dependence and differentiability are all local results.
- Continuous dependence of solution on data for infinite intervals is referred to Lyapunov stability, which is a new branch of ODE.

Homework

1) Suppose that G is open in $R \times R^n$, $f: G \to R^n$ is continuous and local Lipschitz. Show by contradiction method that for any compact set $K \subset G$ there exists $L > 0$ s.t.

$$\| f(t, y) - f(t, x) \| \le L \| x - y \|, \text{ for all } (t, y),\ (t, x) \in K.$$

2) Do the Homework-1, 2, 3 indicated in Lecture 4.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